INFINITE SERIES

BY

BARBARA LEE

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One of the important concepts usually found in a third semester calculus course is infinite series. Although this concept may not appear at first glance to be very important, its consequences have great significance. For example, series are often used as a basis for more advanced mathematics courses, such as differential equations.

Another important consequence involves numerical computations. Many functions such as e, e^x, log x, sin x, cos x, etc. can be written in the form of an infinite series. Using the series form, it is easy to compute a numerical value for the series; and because the value of the series is equivalent to the value of the function, it is easy to find the value of the function--even when a handbook of tables is not available.

A third application of series involves finding the area under a given curve. There are many instances where a methematician, engineer, etc, may need to know if the area under a curve approaches a single fixed value (known as a limit), or if the area does not have a limit, i.e. its area increases indefinitely. There are numerous methods and tests available for determining the above information. Because I am primarily interested in the fundamental concepts of series and the applications of series, I will not present any proofs for the theorems presented. These proofs can be found in any book dealing with calculus, series, and similar subjects.

In order to better understand series, it is helpful to know something about sequences. A sequence is a set of numbers written in an ordered occurrence such as $a_1, a_2, a_3, \cdots$. If a sequence has a fixed number of terms such as 1, 2, 3; the sequence is said to be a finite sequence. If, on the other hand, the sequence has an indefinite number of terms, such as 1, 2, 3, 4, •••; the sequence is said to be an infinite sequence. If the terms of a sequence are such that $a_{n+1} \geq a_n$ for all values of n, the sequence is called a monotonic increasing sequence. Similarly, if for every term in the sequence $a_n \geq a_{n+1}$, the sequence is referred to as a monotonic decreasing sequence. The sequence 1, 2, 3, ••• is an example of a monotonic increasing sequence. An example of a monotonic decreasing sequence is 1, 1/2, 1/3, •••. From these two examples it is also possible to get an idea of what the terms upper bound and lower bound mean. The upper bound of a sequence is the term which is greater than or equal to every other term in the sequence. Similarly, the lower bound of a sequence is the term which is less than or equal to every term of the sequence. From these definitions it is easy to see that in the example given for the monotonic increasing sequence, the lower bound is one and the upper bound as $n \rightarrow$ infinity (∞) becomes infinity. In the same manner for the example of the monotonic decreasing sequence, the upper bound is one and the lower bound as $n \to \infty$ is zero. There are two very important theorems involving monotonic sequences and bounds. One is

> "If (α_n) is a monotonic increasing sequence, then either it has a finite upper bound U, and $\alpha_n \to U$, or its upper bound is $+\infty$, and $\alpha_n \to +\infty$." [1]

The second theorem is

> "If (α_n) is monotonic decreasing, then either it has a finite lower bound L, and $\alpha_n \to L$, or its lower bound is $-\infty$, and $\alpha_n \to -\infty$." [2]

Two other concepts frequently associated with sequences are convergence and divergence. A sequence $a_1, a_2, a_3, \cdots, a_n, \cdots$ is said to converge if for each consecutive value of n, the term a_n approaches an arbitrary but fixed constant 1 without ever surpassing 1. This idea can be rephrased in other ways.

> "The sequence $\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots$ is said to converge, or tend, to zero (in symbols, $\alpha_n \to 0$) if $|\alpha_n|$ can be made as small as we please by taking any n that is sufficiently large." [3]

Or it can be stated as

> "The sequence $\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots$ is said to converge or tend to 1 (in symbols, $\alpha_n \to 1$) if the sequence $\alpha_1 - 1, \alpha_2 - 1, \cdots, \alpha_n - 1, \cdots$ tends to zero." [4]

However, if the term α_n in the sequence $\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots$ increases indefinitely, the series is said to diverge. Therefore, according to these definitions a sequence such as 1/2, 1/6, 1/12, 1/20, ••• converges because as $n \to \infty$, the α_nth term becomes closer and closer to zero. An example for the other type of sequence, i.e. the divergent sequence, is 1, 2, 3, •••, n, •••. In this case as $n \to \infty$ the α_nth term also approaches infinity. Occasionally there are sequences such as 0, 1, 0, 1, ••• which neither converge nor diverge. These sequences are known as oscillating sequences.

These same basic definitions and concepts involving sequences can also be applied to infinite series. An infinite series is an expression which can be written in the form of $a_1 + a_2 + \cdots + a_n + \cdots$, or it can be written more simply as $\sum_{n=1}^{\infty} a_n$. In a manner similar to that of sequences, a series can be classified as a monotonic increasing or a monotonic decreasing series. The concepts of convergence and divergence in a series are also comparable to the concepts as defined for sequences.

> "Suppose we are given any infinite series
> $u_1 + u_2 + u_3 + \cdots$. (2)
> Let $S_n \equiv u_1 + u_2 + \cdots + u_n + \cdots$. We say that the series (2) converges if S_n, as n becomes large, approaches some definite number S. THIS NUMBER S IS CALLED THE SUM OF THE SERIES, the series itself is said to converge, or to be convergent." [5]

Next, let us consider the series $1 + 2 + 3 + \cdots + n + \cdots$. $S_n \equiv u_1 + u_2 + u_3 + \cdots + u_n = 1 + 2 + 3 + \cdots + n$. In this example, if n is large, then so is S_n. The sequence S_n increases indefinitely, that is to say, whatever positive number A we care to put down, S_n will exceed A if we choose n large enough. In a series such as this, the series is said to diverge or to be divergent.

There are many different kinds of series, and because the knowledge of whether or not a series converges or diverges is very important, numerous methods and tests have been devised to obtain this information. One important type of series is known as the geometric series. This is a series written in the form of $1 + r + r^2 + r^3 + \cdots + r^n + \cdots$. Without going into a proof which can be found in any book dealing with series, it has been proven that the series is: (1) convergent when $-1 < r < 1$ and the sum is $(1-r)^{-1}$; (2) divergent when $r \geq 1$; and (3) oscillates between $+\infty$ and $-\infty$ when $r \leq -1$ (in this case, the series is said to be improperly divergent). For example, consider the two following series. Suppose $S_n = 1 + 2 + 4 + \cdots$. In this series, $r = 2$ and the series is divergent. However, in a series such as $S_n = 1 + 1/2 + 1/4 + \cdots$, the ration $r = 1/2$. Therefore, the series is divergent.

A second important series is known as the harmonic series. This series can be written as $1 + 1/2 + 1/3 + 1/4 + \cdots + 1/n$ or it can be written as $\sum_{n=1}^{\infty} 1/n$. Upon first looking at this series, it appears to be a convergent series; but in reality it is a divergent series. Suppose the series is grouped in the following manner: $1 + 1/2 + (1/3 + 1/4) + (1/5 + 1/6 + 1/7 + 1/8) + (1/9 + 1/10 + \cdots + 1/16) + \cdots$. In each case the numerical value of the set of fractions enclosed within each set of parenthesis is greater than 1/2. As a result, by consistently adding these values which are greater than 1/2, the series can be seen to be a divergent series.

A more general form of the harmonic series can be written as $\sum_{n=1}^{\infty} 1/n^k$, which is called the hyperharmonic series. (This series becomes the harmonic series when $k = 1$). In a manner similar to that of the harmonic series, this series may be grouped as $1 + (1/2^k + 1/3^k) + (1/4^k + 1/5^k + 1/6^k + 1/7^k) + (1/8^k + 1/9^k + \cdots + 1/15^k) + \cdots$ if $k > 1$. Then we can rewrite each group contained within the parenthesis. $1/2^k + 1/3^k < 1/2^k + 1/2^k = 2/2^k \doteq 1/2^{k-1}$. In the same manner $1/4^k + 1/5^k + 1/6^k + 1/7^k < 1/4^k + 1/4^k + 1/4^k + 1/4^k = 4/4^k = 1/(2^{k-1})^2$. By continuing this process the hyperharmonic series can be rewritten as $1 + 1/(2^{k-1}) + 1/(2^{k-1})^2 + 1/(2^{k-1})^3 + \cdots$. From this new notation we can see that the hyperharmonic series is actually a geometric series with $r = 1/(2^{k-1})$ which is less than 1 when $k > 1$. Therefore, the hyperharmonic series $\sum_{n=1}^{\infty} 1/n^k$ is convergent when $k > 1$, and it is properly divergent when $k \leq 1$. This series has also been called the p-series by some mathematicians.

A final series often used in connection with convergence and divergence tests is $\sum_{n=1}^{\infty} 1/n(n+1)$ which is always convergent as $n \to \infty$. Using these various types of series as a basis (geometric, harmonic, hyperharmonic or p-series, and $\sum_{n=1}^{\infty} 1/n(n+1)$), one is able to determine whether or not nay given series converges or diverges.

One of the easiest means of testing a given series is to compare it with a particular series which is already known to specifically converge or diverge. For example, if k is any fixed but arbitrary constant and $\sum_{n=1}^{\infty} c_n$ is known to be a convergent series, then the series $\sum_{n=1}^{\infty} kc_n$ also converges and its limit is $k \sum_{n=1}^{\infty} c_n$. Similarly, if $\sum_{n=1}^{\infty} d_n$ is known to be a divergent series, then the series $\sum_{n=1}^{\infty} kd_n$ also diverges (provided $k \neq 0$) and its limit is $k \sum_{n=1}^{\infty} d_n$.

Now suppose one has two series having positive terms $\sum_{n=1}^{\infty} c_n$ and $\sum_{n=1}^{\infty} d_n$, where it is known that the first one, $\sum_{n=1}^{\infty} c_n$, converges and the second one, $\sum_{n=1}^{\infty} d_n$, diverges. Then for any series of positive terms $\sum_{n=1}^{\infty} a_n$ such that $a_n \leq c_n$ for all values of n, the given series $\sum_{n=1}^{\infty} a_n$ also converges. If, however, $a_n \geq d_n$, for all values of n, the series $\sum_{n=1}^{\infty} a_n$ also diverges. An example of the latter case is $\sum_{n=1}^{\infty} 1/\sqrt{n}$. For every term $1/\sqrt{n} > 1/n$. Since $\sum_{n=1}^{\infty} 1/n$ is known to be divergent, the series $\sum_{n=1}^{\infty} 1/\sqrt{n}$ is also divergent.

There is also another comparison test which is used frequently.

> "If $\sum_{n=1}^{\infty} u_n$, $\sum_{n=1}^{\infty} v_n$ are two series of positive terms such that $u_n/v_n \to L > 0$, then the two series are either both convergent or both divergent. The number L must be finite and not zero in all applications of this theorem." [6]

Let $\sum_{n=1}^{\infty} c_n$ be a convergent series and $\sum_{n=1}^{\infty} a_n$ be a series to be tested. "If $a_n/c_n < k$(a finite constant) for every n, then $\sum_{n=1}^{\infty} a_n$ is convergent."[7] Suppose one is given the series $\sum_{n=1}^{\infty} 1/n(n+2)$ and asked to find out whether or not the series converges. Then using the above theorem: $\frac{1/n(n+2)}{1/n^2} = \frac{n^2}{n(n+2)}$ $= \frac{n}{(n+2)}$ which is always less than 1. Since the series must either both be convergent or both be divergent, and we know that $\sum_{n=1}^{\infty} 1/n^2$ converges, the series $\sum_{n=1}^{\infty} 1/n(n+2)$ must also converge.

"If $a_n/d_n > k$(a finite constant) for every n, then $\sum_{n=1}^{\infty} a_n$ is divergent."[8] Suppose one is given the series $\sum_{n=1}^{\infty} \frac{(n+3)}{n(n+2)}$ and asked to find out whether it is convergent or divergent. $\frac{(n+3)/n(n+2)}{1/n} = \frac{n(n+3)}{n(n+2)} = \frac{(n+3)}{(n+2)}$ which is always greater than one. Using the same reasoning as in the previous example and knowing that the series $\sum_{n=1}^{\infty} 1/n$ diverges, we can say that the series $\sum_{n=1}^{\infty} \frac{(n+3)}{n(n+2)}$ also diverges.

There is still another comparison test which involves $\sum_{n=1}^{\infty} c_n$ and $\sum_{n=1}^{\infty} d_n$ which are a convergent series and a divergent series respectively. Suppose the series to be tested is $\sum_{n=1}^{\infty} a_n$. All three of these series are comprised of positive terms. If $\frac{a_{n+1}}{a_n} \leq \frac{c_{n+1}}{c_n}$ then the series $\sum_{n=1}^{\infty} a_n$ converges. And if $\frac{a_{n+1}}{a_n} \geq \frac{d_{n+1}}{d_n}$ then the series

$\sum_{n=1}^{\infty} a_n$ diverges.

One frequently used test is the ratio test. In this test, let $a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ be a series of positive terms, and let $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = k$, where k is a constant. If $k < 1$ the series is convergent. If $k > 1$ the series is divergent, and if $k = 1$ the test is inconclusive. An example using this test could be to decide whether or not the series $\sum_{n=1}^{\infty} \frac{n^{20}}{2^n}$ converges. $\frac{a_{n+1}}{a_n} = \frac{(n+1)^{20}}{2^{n+1}} \div \frac{n^{20}}{2^n}$ $= \frac{1}{2}\left(\frac{n+1}{n}\right)^{20} = \frac{1}{2}\left(1+\frac{1}{n}\right)^{20}$ and the $\lim_{n\to\infty} \frac{1}{2}\left(1+\frac{1}{n}\right)^{20} = \frac{1}{2}$. Therefore, by the ratio test, the series converges.

A test very similar to the ratio test is D'Alembert's ratio test.

> "If $\frac{a_{n+1}}{a_n} < r < 1$ for every n, then $\sum_{n=1}^{\infty} a_n$ is convergent, and if $\frac{a_{n+1}}{a_n} \geq 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent." 9

An example of this method could be used for the following series. Suppose A is a series such that

$$A = \frac{3}{5} + \frac{3(5)}{5(10)} + \frac{3(5)(7)}{5(10)(15)} + \cdots + \frac{3(5)(7)\cdots(2n+1)}{5(10)(15)\cdots(5n)}.$$

$\frac{a_{n+1}}{a_n} = \frac{2n+1}{5n} = \frac{2+1/n}{5}$. As n approaches infinity this ratio will approach 2/5, but it will always be less than 1. Therefore, the series is convergent.

A test which is closely related to D'Alembert's ratio test is Raabe's test.

> "$\sum_{n=1}^{\infty} u_n$ is a series of positive terms: if $n\left(\frac{u_n}{u_{n+1}} - 1\right) \to l > 1$, then $\sum_{n=1}^{\infty} u_n$ is convergent;

if n $\left(\frac{u_n}{u_{n+1}} - 1\right) \to l < 1$, then $\sum_{n=1}^{\infty} u_n$ is divergent." 10

When a series is written in the form of a radical, the most useful test is often Cauchy's root test.

"If u_n is always positive and $\sqrt[n]{u_n} \to L$, then the series $\sum_{n=1}^{\infty} u_n$ is convergent when $L < 1$, and divergent when $L > 1$." 11

If one is given a series such as $\sum_{n=1}^{\infty} 1/n^n$ to be tested, Cauchy's root test would be very useful. $\sqrt[n]{\frac{1}{n^n}} = \frac{1}{n}$. As n becomes large, this ratio approaches zero. Therefore, the series is convergent.

Another frequently used test, and one which is quite different from the previously mentioned tests, is the integral test. For this test let f(x) be a continuous, positive, monotonically decreasing function defined for all real numbers $x \geq 1$, and let $a_n = f(n)$ for all positive integers n. Then the infinite series $\sum_{n=1}^{\infty} a_n$ converges if and only if the improper integral $\int_1^{\infty} f(x)dx$ converges. If one is asked to discover if a series such as $\frac{1}{1(2)} + \frac{1}{2(3)} + \frac{1}{3(4)} + \ldots$ converges, one could very easily use the integral test. $\lim_{t \to \infty} \int_1^t \frac{dx}{x(x+1)}$ $= \lim_{t \to \infty} \left[\log \frac{x}{x+1}\right]_1^t = \lim_{t \to \infty} \left[\log \frac{t}{t+1} - \log \frac{1}{2}\right] =$ $\log 1 - \log \frac{1}{2} = -\log \frac{1}{2}$. Therefore, the series converges.

In reading about infinite series one will eventually find a statement such as--this series converges absolutely. An infinite series is said to converge absolutely if $\sum_{n=1}^{\infty} a_n = |a_1| + |a_2| + |a_3| + |a_4| + \cdots + |a_n| + \cdots$ converges. If a series is absolutely convergent, then it is said to converge. But if a series is convergent, it does not necessarily mean that the series is absolutely convergent. When a series is convergent but it is not absolutely convergent, then the series formed by its positive or negative terms alone is divergent. For example the series $1 - 1/2 + 1/3 - 1/4 + \cdots$ converges. However when this same series is comprised of only positive terms, it becomes $1 + 1/2 + 1/3 + 1/4 + \cdots$, and this series is already known to be a divergent series.

If a series is absolutely convergent, it is possible to rearrange the terms of the series without causing any difference in whether or not the series converges. This occurs because no matter how the terms are added, or in what order they are added, the sum will always approach the same finite limit. However, if the series is non-absolutely convergent, then the rearrangement of the terms will make a difference. For example suppose that you are given an alternating series such as $1 - 1/2 + 1/3 - 1/4 + \cdots$. (An alternating series is a series in which every positive term is consistently followed by a negative term, and every negative term is consistently followed by a positive term. Therefore, the series may be written as $a_1 - a_2 + a_3 - a_4 + \cdots$.) The sum of the series $1 - 1/2 + 1/3 - 1/4 + \cdots$ is log 2. However, if this series is rearranged such that there are always two positive terms followed by a single negative term, then the series will become $1 + 1/3 - 1/2 + 1/5 + 1/7 - 1/4 + \cdots$. The sum of this series is 3/2 log 2. Therefore, if the series is non-absolutely convergent, rearrangement of the terms will make a difference in the sum of the series.

Dirichlet's test is often used by mathematicians when absolute convergence is involved.

> "If (a_n) is a sequence of numbers such that, for some fixed number K, $|a_1 + a_2 + \cdots + a_n| < K$ for all n, and (v_n) is a monotonic sequence that converges to zero, then $\sum_{n=1}^{\infty} a_n v_n$ is convergent." [12]

One important aspect of series involves the idea of improper integrals. This is used most frequently in finding the area under a given curve.

$$\int_a^{\infty} f(x)dx = \lim_{b \to \infty} \int_a^b f(x)dx$$

$$\int_{-\infty}^{b} f(x)dx = \lim_{a \to -\infty} \int_a^b f(x)dx$$

and $$\int_{-\infty}^{+\infty} f(x)dx = \lim_{a \to -\infty} \int_a^0 f(x)dx + \lim_{b \to +\infty} \int_0^b f(x)dx.$$

When the limit exists, the series converges. When the limit does not exist, the series diverges.

One common test for integrable functions is the comparison test.

"If f(x) and g(x) are integrable functions

> when $x \geq a$ and $0 \leq f(x) \leq g(x)$, then $\int_a^\infty f(x)dx$ converges if $\int_a^\infty g(x)dx$ converges, $\int_a^\infty g(x)dx$ diverges if $\int_a^\infty f(x)dx$ diverges." 13

If one is given the series $\sum_{n=1}^{\infty} \frac{dx}{(1+x^4)^{\frac{1}{2}}}$ to test

for convergence, $\int_1^\infty \frac{dx}{(1+x^4)^{\frac{1}{2}}} < \int_1^\infty \frac{dx}{(x^4)^{\frac{1}{2}}} = \int_1^\infty \frac{dx}{x^2}$

$= \lim_{b\to\infty} \int_1^b \frac{dx}{x^2} = 1.$

> "If $f(x)$ and $g(x)$ are non-negative integrable functions when $x \geq a$ and $\lim_{x\to\infty} \frac{f(x)}{g(x)} = c \neq 0$ exist, then either both integrals $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ converge or both diverge. If $c=0$ and $\int_a^\infty g(x)dx$ converges, then $\int_a^\infty f(x)dx$ also converge. If $c=\infty$ and $\int_a^\infty g(x)dx$ diverges, then $\int_a^\infty f(x)dx$ also diverges." 14

Two similar theorems exist when $f(x)$ and $g(x)$ are integrable functions bounded by a finite interval $[a,b]$.

> "If $f(x)$ and $g(x)$ are integrable on $a < x \leq b$ (or $a \leq x < b$) and such that $0 \leq f(x) \leq g(x)$, then $\int_a^b f(x)dx$ converges if $\int_a^b g(x)dx$ converges, $\int_a^b g(x)dx$ diverges if $\int_a^b f(x)dx$ diverges." 15

> "If $f(x)$ and $g(x)$ are positive integrable functions on $a < x \leq b$ and if $\lim_{x\to a} \frac{f(x)}{g(x)} = c \neq 0$ exists, then the integrals $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ both converge or both diverge. If $\lim_{x\to a^+} \frac{f(x)}{g(x)} = 0$ and $\int_a^b g(x)dx$ converges, then $\int_a^b f(x)dx$ converges. If $\lim_{x\to a^+} \frac{f(x)}{g(x)} = \infty$ and $\int_a^b g(x)dx$ diverges, then $\int_a^b f(x)dx$ diverges." 16

One of the most important kinds of series is the power series. This is a series which can be written in the form $f(x) = \sum_{n=0}^{\infty} a_n(x-b)^n$, which is a power series in (x-b). Some of the common examples are $\sum_{n=1}^{\infty} n^n x^n$ which converges only when $x = 0$; $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ which converges for all values of x; and $\sum_{n=1}^{\infty} \frac{x^n}{n}$ which converges when $|x| < 1$, diverges when $|x| > 1$, and in inconclusive when $|x| = 1$.

Suppose $f(x) = a_0 + a_1(x-b) + a_2(x-b)^2 + \cdots + a_n(x-b)^n$. Then $f'(x) = a_1 + 2a_2(x-b) + 3a_3(x-b)^2 + \cdots$. $f''(x) = 2a_2 + 2(3)a_3(x-b) + 3(4)a_4(x-b)^2 + \cdots$. $f^{(n)}(x) = n!a_n + (n+1)!a_{n+1}(x-b) + \frac{(n+2)!}{2!}(x-b)^2 + \cdots$. If we let $x = b$, then $f(b) = a_0$, $f'(b) = a_1$, $f''(b) = 2!a_2$, $\cdots$, $f^{(n)}(x) = n!a_n$. In this manner we can rewrite the original power series as $f(x) = f(b) + f'(b)(x-b) + \frac{f''(b)}{2!}(x-b)^2 + \frac{f'''(b)}{3!}(x-b)^3 + \cdots + \frac{f^{(n)}(b)}{n!}(x-b)^n$. This series is commonly known as the Taylor's series. In the special case where $b = 0$, the Taylor's series becomes $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$.

This special case is called the Maclaurin's series. These two series are very important because they form the basic concept behind many of the mathematical computations for particular functions. Several functions can be written in the form of these two series and then values can be obtained even when tables are not available. For example $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$. The series equivalent to $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$. These are just two instances in which knowing infinite series can be very helpful in making numerical computations.

What I have presented in this paper is in itself only a brief introduction into the topic of infinite series. The subject can become much more involved when one gets into such areas as double and multiple series (which resemble matrices in many aspects), matrix or series transformations, etc. I did not go into these concepts and many other concepts because it would make my topic too broad and because these concepts are much more complicated theories which I don't feel most mathematics majors would use again--either in many of their more advanced mathematics courses or, especially, in their future jobs. As a result I have limited myself to general definitions and to basic concepts which I feel every mathematics major should know for his own personal knowledge and possible for use in his future job.

Footnotes

[1]W. L. Farrar, A Text-book of Convergence (London: Oxford University Press, 1938), p. 18.

[2]Ibid., p. 19.

[3]Ibid., p. 8.

[4]Ibid., p. 10.

[5]Ibid., p. 4.

[6]Ibid., p. 29.

[7]Lloyd L. Smail, Elements of the Theory of Infinite Processes (New York: McGraw-Hill book Co., Inc., 1923), p. 75.

[8]Ibid.

[9]Ibid., p. 78.

[10]W. L. Farrar, A Text-book of Convergence (London: Oxford University Press, 1938), p. 30.

[11]Ibid., p. 44.

[12]Ibid., p. 63.

[13]O.E. Stanaitis, An Introduction to Sequences, Series, and Improper Integrals. (San Francisco: Holden-Day, Inc., 1967), p. 157.

[14]Ibid.

[15]Ibid., p. 165.

Bibliography

Bromwich, T.J.I'A. *Theory of Infinite Series.* London: Macmillan and Co., Limited, 1949.

Dienes, P. *The Taylor Series: An Introduction to the Theory of Functions of a Complex Variable.* Oxford: Clarendon Press, 1931.

Farrar, W.L. *A Text-book of Convergence.* London: Oxford University Press, 1938.

Knopp, Dr. Konrad. *Theory and Application of Infinite Series.* New York: Hafner Publishing Co., 1948.

Purcell, Edwin J. *Calculus with Analytic Geometry.* New York: Appleton-Century-Crofts, Division of Meredith Publishing Co., 1965.

Smail, Lloyd L. *Elements of the Theory of Infinite Processes.* New York: McGraw-Hill Book Co., Inc., 1923.

Stanaitis, O.E. *An Introduction to Sequences, Series, and Improper Integrals.* San Francisco: Holden-Day, Inc., 1967.